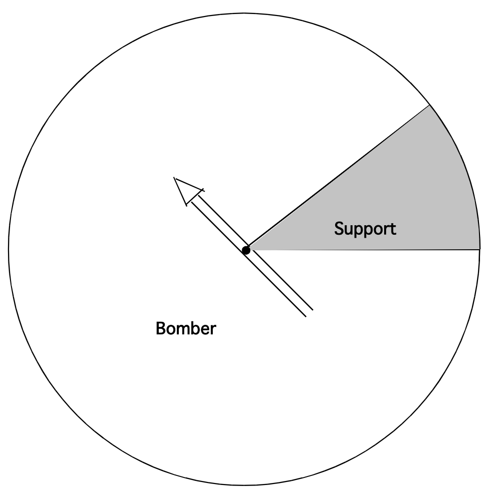
Mixed Strategy

Example Recall the case of General Roadrunner and General Coyote; General Roadrunner sends out a bombing sortie consisting of a heavily armed bomber plane and a lighter support plane. every day. He can place the bomb on either plane and General Coyote chooses to attack one of the planes. The payoffs shown in the matrix below give the percentage of time the bomb hits the target in each of the four possible situations.

		C. attacks	
		B	S
R.	B	80%	100%
places bomb	S	90%	50%

Last day, we saw that this payoff matrix **does not have a saddle point.** If General Roadrunner chooses a fixed strategy of placing the bomb on the bomber each day, General Coyote can reduce the percentage of time the bomb hits the target to 80% by attacking the bomber each day. Therefore General Roadrunner might want to bluff sometimes and put the bomb on the support plane. This would confuse General Coyote who would now have to decide whether to attack the support plane or the bomber plane. The mathematical model for this type of situation assumes that General Roadrunner does something like putting the bomb on the bomber 90% of the time and on the support plane 10% of the time. He makes his decision of where to put the bomb on each play by using a device such as a spinner like the one shown below, so that his decisions are not predictable. General Coyote, who will use relative frequencies to determine General Roadrunner's strategy, is then faced with the task of coming up with the best counterstrategy (we will discuss a best counterstrategy in more detail when we master calculating with mixed strategies).



Mixed Strategies (How to represent it mathematically). For a zero-sum game with pay-off matrix for R given by

$$\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$$

R plays $\begin{bmatrix} .8 & .2 \end{bmatrix}$, means that R plays $R1$ with probability .8 and R plays $R2$ with probability .2. C plays $\begin{bmatrix} .3 \\ .7 \end{bmatrix}$ means that C plays $C1$ with probability .3 and C plays $C2$ with probability .7.

NOTE: We assume that R and C play the game repeatedly and that on each play both can choose their row or column with the aid of some random device so that their choice is unpredictable. They might use a random spinner such as the one shown above, with the appropriate markings or alternatively they might roll a die to make their choice of strategy on each play. Although neither player can predict what their opponent will do on the next play, they can deduce from the relative frequency of previous plays what the mixed strategy for their opponent is and respond accordingly.

If we let X denote R's payoff for this game, we see that on each play, the random variable X has 4 possible outcomes. We can calculate the expected value of X using the probability distribution of X. For this game, if R plays $\begin{bmatrix} .8 & .2 \end{bmatrix}$ and C plays $\begin{bmatrix} .3 \\ .7 \end{bmatrix}$ the probability distribution for X is given by the following table.

Choice	X = Pay-off for R	Probability	$XP(X)$
$R1C1$	-1	$(.8)(.3) = .24$	$-1 \times .24$
$R1C2$	3	$(.8)(.7) = .56$	$3 \times .56$
$R2C1$	2	$(.2)(.3) = .06$	$2 \times .06$
$R2C2$	-2	$(.2)(.7) = .14$	$-2 \times .14$
			$\mu = 1.28$

Note that in calculating the probabilities here we are assuming that the **players' choices are independent of each other**. So neither player has knowledge of what the other player is choosing. This is the reason that we multiply the probabilities. For example in calculating the probability that R will play $R1$ and C will play $C1$ (and R will lose \$1, a pay-off of -1), we multiply .8, the probability that R will play $R1$, by .2, the probability that C will play $C1$.

The expected pay-off for R is 1.28. This gives the average pay-off for R, if R and C play this game many times with the given strategies. Because it is a zero-sum game the **expected pay-off for C** and the expected pay-off for R must add to zero, since R's gain is always C's loss. So **C's expected pay-off here is -1.28.**

As it turns out, the expected pay-off can be calculated using matrix multiplication. **to calculate the Expected Pay-off for** R for the above game, if R plays $\begin{bmatrix} .8 & .2 \end{bmatrix}$ and C plays $\begin{bmatrix} .3 \\ .7 \end{bmatrix}$ we multiply matrices as follows:

$$\begin{bmatrix} .8 & .2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} .3 \\ .7 \end{bmatrix} = (1.28)$$

Calculating the expected payoff

Example Suppose General Roadrunner puts the bomb on the bomber 90% of the time and on the support plane 10% of the time. Suppose also that General Coyote attacks the bomber 50% of the time and attacks the support plane 50% of the time. What is the expected percentage of successful bombing missions?

$$\begin{bmatrix} 80 & 100 \\ 90 & 50 \end{bmatrix}$$

Note Because this is a constant sum game, the expected payoff for C is $100 - \nu$, where ν = expected payoff for R.

Example R and C play a game with Pay-off matrix for R given by

$$\begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix}.$$

If R plays a strategy $\begin{bmatrix} .4 & .6 \end{bmatrix}$ and C plays a strategy $\begin{bmatrix} .2 \\ .8 \end{bmatrix}$,

(a) what is the expected payoff for R?

(b) What is the expected payoff for C?

(c) (Different Counterstrategy) If C changes strategy to $\begin{bmatrix} .3 \\ .7 \end{bmatrix}$, what happens to C's expected pay-off?

Note:

- The expected payoff for R (ν) is the negative of the expected payoff for C for zero sum games or $K - \nu$ for constant sum games with constant K.
- The expected payoff for R varies as C's strategy varies.

The General Case

Consider the general situation where a zero-sum game has a pay-off matrix for R given by a matrix A with m rows (R has m options) and n columns (C has n options). We can represent R's strategy by a row matrix with m entries

$$[r_1, r_2, r_3, \ldots, r_m],$$

where r_i gives the probability that R will play option Ri (Row i), and

$$0 \leq r_i \leq 1, \qquad r_1 + r_2 + r_3 + \cdots + r_m = 1.$$

We can represent C's strategy by a column matrix:

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

where c_i gives the probability that C will play option Ci (Col i), and

$$0 \leq c_i \leq 1, \qquad c_1 + c_2 + c_3 + \cdots + c_n = 1.$$

Then the **Expected pay-off for** R **(**$=\mu$ **)** is given by

$$\begin{bmatrix} r_1, r_2, \ldots, r_m \end{bmatrix} A_{m \times n} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = (\mu)$$

Example If the payoff matrix for R, for a zero-sum game, is given by

$$\begin{bmatrix} 2 & -3 \\ 0 & 2 \\ -5 & 10 \end{bmatrix},$$

and R's strategy is given by [.2 .5 .3], and C's strategy is given by $\begin{bmatrix} .7 \\ .3 \end{bmatrix}$.

(a) What is the expected pay-off for R for this game?

(b) What is the expected pay-off for C for this game?

Example (Two Finger Morra) Ruth and Charlie play a game. At each play, Ruth and Charlie simultaneously extend either one or two fingers and call out a number. The player whose call equals the total number of extended fingers wins that many pennies from the opponent. In the event that neither player' call matches the total, no money changes hands.

The pay-off matrix for this game is given by:

$$\begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix}$$

(a) If Ruth chooses all of her options with equal probability and Charlie chooses each of his four options with equal probability, what are s the expected winnings for Ruth?

(b) If Ruth chooses all of her options with equal probability and Charlie sticks to a pure strategy of playing Col. 2 on every play, what are the expected winnings for Ruth?

(c) If Ruth always chooses all of her options with equal probability, which of the above two strategies is the better one for Charlie?

Old Exam Questions

1 Rasputin (R) and Catherine (C) play a zero-sum game with payoff matrix for Rasputin given below. If Rasputin's strategy is given by $[.3, .2, .5]$ and Catherine's strategy is given by $\begin{bmatrix} .2 \\ .1 \\ .7 \end{bmatrix}$, what is the expected pay-off for Rasputin?

	C_1	C_2	C_3
R_1	1	0	0
R_2	1	5	0
R_3	3	2	-2

(a) -0.1 (b) 1 (c) 1.54 (d) -0.3 (e) 0.21

2 Rusty (R) and Crusty (C) play a zero-sum game with pay-off matrix for Rusty given by:

	$C1$	$C2$
$R1$	2	1
$R2$	5	-1

If Crusty plays C2 on every play, which of the following mixed strategies gives the highest expected pay-off for Rusty?

(a) (.2 .8) (b) (1 0) (c) (0 1) (d) (.5 .5) (e) (.7 .3)